Continued Fractions, Diophantine Approximations, and Design of Color Transforms

Yuriy Reznik

Qualcomm Incorporated

SPIE Applications of Digital Image Processing, San Diego, CA, August 2008.

Outline

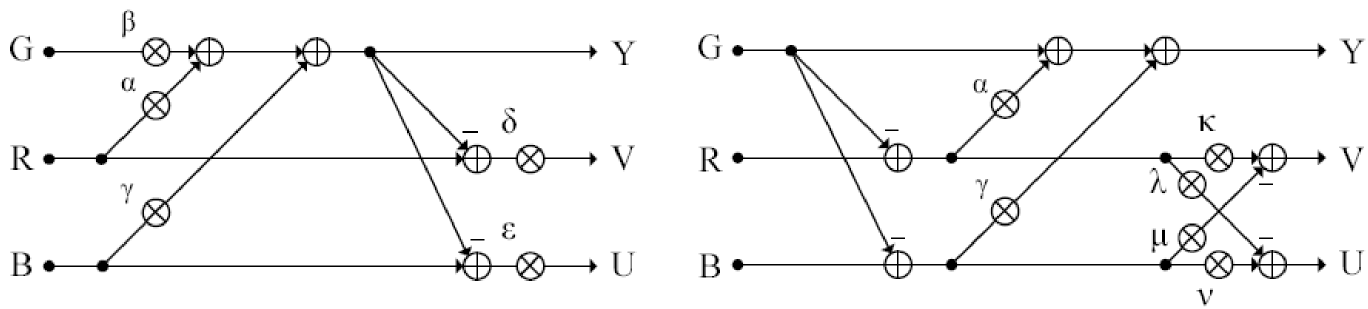
- Color transforms –
  - basics and design examples
- The idea and the problem
- Some useful facts
  - continued fractions, convergents,
  - simultaneous Diophantine approximations
- Connection to our problem
- Precision bounds
- Examples of transform designs

Color transforms

- Fundamental operations in image acquisition, processing, coding/decoding, reproduction
- Most frequently – a connection between
  - Reference color space (CIE RGB, XYZ, etc)
  - System/device-specific space
    - YUV, YIQ, YDbDr – color television systems
    - YCbCr (ITU-R BR.601, 705) – digital video
    - CMYK, CMYKOG, etc. – printers
- In many cases:
  - a linear transform with matrix of constant factors

Examples of Color Transforms

- RGB to YUV-type transforms:



Factors	Color spaces / Standards					
	YUV PAL	**YDbDr** SECAM	**YPbPr/YCbCr** ITU-R BT.601	**YPbPr/YCbCr** ITU-R BT.709	**YIQ** NTSC	**YSrSb** [6]
α	0.299	0.299	0.299	0.2125	0.299	0.3227
β	0.587	0.587	0.587	0.7154	0.587	0.3447
γ	0.114	0.114	0.114	0.0721	0.114	0.3326
δ	$0.615\,\frac{1}{1-\alpha}$	$1.333\,\frac{1}{1-\alpha}$	$\frac{1}{2}\frac{1}{1-\alpha}$	$\frac{1}{2}\frac{1}{1-\alpha}$		
ϵ	$0.436\,\frac{1}{1-\gamma}$	$-1.333\,\frac{1}{1-\gamma}$	$\frac{1}{2}\frac{1}{1-\gamma}$	$\frac{1}{2}\frac{1}{1-\gamma}$		
κ	0.615	1.333	$\frac{1}{2}$	$\frac{1}{2}$	$0.877\,(1-\alpha)\cos(33)$ $+0.492\,\alpha\,\sin(33)$	-0.1643
λ	$0.436\,\frac{\alpha}{1-\gamma}$	$-1.333\,\frac{\alpha}{1-\gamma}$	$\frac{1}{2}\frac{\alpha}{1-\gamma}$	$\frac{1}{2}\frac{\alpha}{1-\gamma}$	$-0.877\,(1-\alpha)\sin(33)$ $+0.492\,\alpha\,\cos(33)$	0.5095
μ	$0.615\,\frac{\gamma}{1-\alpha}$	$1.333\,\frac{\gamma}{1-\alpha}$	$\frac{1}{2}\frac{\gamma}{1-\alpha}$	$\frac{1}{2}\frac{\gamma}{1-\alpha}$	$0.877\,\gamma\,\cos(33)$ $+0.492\,(1-\gamma)\sin(33)$	0.3470
ν	0.436	-1.333	$\frac{1}{2}$	$\frac{1}{2}$	$-0.877\,\gamma\,\sin(33)$ $+0.492\,(1-\gamma)\cos(33)$	0.3870

Implementations of transforms

- Specification:
  - Y= 0.299 R + 0.587 G + 0.114 B
- Engineering folklore (cf. wikipedia):
  - Y’ = [77 R + 108 G + 21 B] / 256
  - which maps to integer instructions:
    - Y’ = (77*R + 108*G + 21*B + 128) >> 8;
    - * – multiply, >> – shift right operator
- Consequences:
  - errors: |0.299 – 77/256| = 0.00178125
  - resources: needs at least 8 bits of bandwidth

The problem and the idea

- Problem:
  - Given real (irrational) constants $\theta_1, \ldots, \theta_m,\ m \geqslant 2$
  - We need to improve precision of their dyadic approximations:

$$\theta_1 \approx p_1/2^k\,,\ \ldots\ ,\ \theta_m \approx p_m/2^k$$

  - while keeping the number of "mantissa" bits *k* small !
- Idea:
  - Introduce some additional factor ξ
  - then find dyadic approximations:

$$\theta_1\xi \approx p_1/2^k\,,\ \ldots\ ,\ \theta_m\xi \approx p_m/2^k$$

  - and then apply the inverse factor $1/\xi$ elsewhere

Preview of main result

- Normal dyadic approximations:
  - best choice of integers gives

$$\left|\theta_i - p_i/2^k\right| = 2^{-k}\left|2^k\theta_i - p_i\right| = 2^{-k}\min_{z\in\mathbb{Z}}\left|2^k\theta_i - z\right| \leqslant 2^{-k-1}$$

  - hence

$$\Delta(k) = \min_{p_1,\ldots,p_m}\max_i\left\{\left|\theta_i - p_i/2^k\right|\right\} \leqslant 2^{-k-1}$$

- Scaled dyadic approximations:
  - normalized error:

$$\Delta_\xi(k) = \frac{1}{\xi}\min_{p_1,\ldots,p_m}\max_i\left\{\left|\theta_i\xi - p_i/2^k\right|\right\} \lesssim 2^{-k\left(1+\frac{1}{m-1}\right)}$$

  - E.g., when m=2, we need just half the bits (*k*) to achieve same precision!!!

Explanation of main result

- Things to follow:
  - Review of some basic facts
    - Continued fractions
    - Convergents
    - Diophantine approximations
    - Simultaneous rational approximations
  - Connection to our problem

Continued fractions

- Continued fraction:

$$[a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots \frac{1}{a_n}}}$$

  - any rational can be presented in this form.
  - Moreover, given any irrational $\theta = \theta_0$
    - we can produce series

$$a_n = \lfloor \theta_n \rfloor, \quad \theta_{n+1} = \frac{1}{\theta_n - a_n}, \quad n = 1, 2, \ldots$$

    - such that $\lim_{n \to \infty} [a_0, a_1, \ldots, a_n] = \theta$

Equivalence, convergents

- Equivalent irrational numbers:

$$\begin{aligned}\theta &= [a_0, a_1, \ldots, a_l, c_1, c_2, \ldots] \\ \theta' &= [b_0, b_1, \ldots, b_m, c_1, c_2, \ldots]\end{aligned}$$

- Convergents:

$$\theta = [a_0, a_1, \ldots] \qquad \frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$$

  - can be recursively computed as follows:

$$\begin{aligned}&p_{-1} = 1, \quad p_0 = a_0, \quad p_n = a_n p_{n-1} + p_{n-2}, \quad n = 1, 2, \ldots \\ &q_{-1} = 0, \quad q_0 = 1, \quad\ \ q_n = a_n q_{n-1} + q_{n-2},\end{aligned}$$

Best rational approximations

- Let:

$$\theta = [a_0, a_1, \ldots] \qquad \frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$$

- Key properties:
  - p_n, q_n are growing exponentially fast
  - they produce approximations with <u>quadratic</u> error decay:

$$|\theta - p_n/q_n| < 1/q_n^2$$

  - such approximations are <u>best</u> in a sense that

$$\left|\theta - p_n/q_n\right| < \left|\theta - p/q\right| \quad \text{for any} \quad p,\ 0 < q < q_n$$

Precision of rational approximations

- More precise statement:

THEOREM 2.1. *Let θ be irrational.*
Then there exist infinitely many integers q and p such that

$$|\theta - p/q| < \kappa(\theta) q^{-2},$$

where:

$$\kappa(\theta) = \left[\begin{array}{lll} \frac{1}{\sqrt{5}}, & \text{if } \theta \text{ equivalent to } \frac{\sqrt{5}-1}{2} & (\text{root of } \theta^2 + \theta - 1 = 0), \\ \frac{1}{2\sqrt{2}}, & \text{if } \theta \text{ equivalent to } \sqrt{2} - 1 & (\text{root of } \theta^2 + 2\theta - 1 = 0), \\ \frac{5}{\sqrt{221}}, & \text{if } \theta \text{ equivalent to } \frac{\sqrt{221}-11}{10} & (\text{root of } 5\theta^2 + 11\theta - 5 = 0), \\ \frac{13}{\sqrt{1517}}, & \text{if } \theta \text{ equivalent to } \frac{\sqrt{1517}-29}{26} & (\text{root of } 13\theta^2 + 29\theta - 13 = 0), \\ \ldots & & \end{array} \right.$$

is a chain producing a sequence $\frac{1}{\sqrt{5}}, \frac{1}{2\sqrt{2}}, \frac{5}{\sqrt{221}}, \frac{13}{\sqrt{1517}}, \ldots$ *that tends to* $\frac{1}{3}$.

Simultaneous approximations

- Precision of simultaneous approximations:

THEOREM 2.2. *Let* $\theta_1, \ldots, \theta_m$, $(m \geqslant 2)$ *be irrationals. Then, there are infinitely many integers* q *and* $p_1, \ldots, p_m$, *such that*

$$\max_i \{|\theta_i - p_i/q|\} < \tfrac{m}{m+1}\, q^{-1-1/m} \,.$$

- Observations:
  - quadratic convergence rate when *m=2*
  - for large *m* convergence rate drops down to linear

Connection to our problem

- Consider:

$$\theta_1 \xi \approx p_1/2^k \,, \quad \ldots \quad , \; \theta_m \xi \approx p_m/2^k$$

- We immediately notice that by setting

$$\xi := q/2^k$$

- and then applying

$$\tfrac{1}{\xi} \left| \xi \, \theta_i - p_i/2^k \right| = \left| \theta_i - p_i/q \right| \,, \quad i = 1, \ldots, n \,,$$

- we arrive at standard m-ary rational approximation problem!

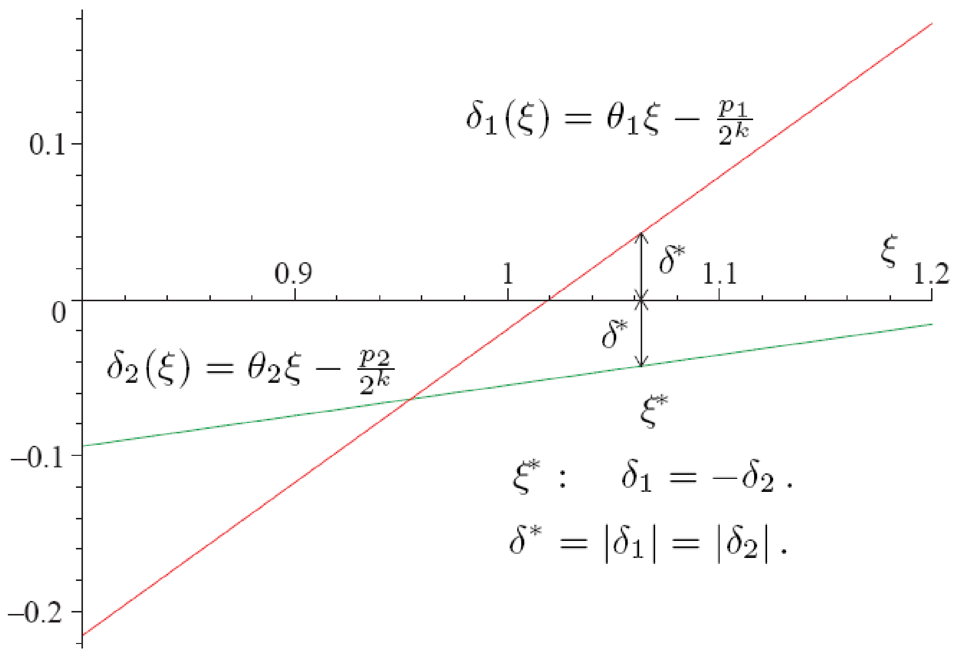
- However… there is more…

Approximation of pair of constants

- Let m=2, and let:

$$\delta_1(\xi) = \theta_1\xi - p_1/2^k\,, \qquad \delta_2(\xi) = \theta_2\xi - p_2/2^k$$

- Let's plot these functions:



Approximations of pairs of constants

- We arrive at:

Lemma 1. *Let θ_1, θ_2 be real numbers, such that $\theta_1\theta_2 > 0$, and let k, p_1, and p_2 be integers. Then, there exist values ξ^* and δ^*, such that*

$$\delta^* = \max\left\{|\delta_1(\xi^*)|, |\delta_2(\xi^*)|\right\} = \min_{\xi} \max\left\{|\delta_1(\xi)|, |\delta_2(\xi)|\right\} .$$

These values are:

$$\xi^* = \tfrac{1}{2^k}\tfrac{p_1+p_2}{\theta_1+\theta_2} , \tag{8}$$

and

$$\delta^* = \tfrac{1}{2^k}\left|\theta_1\tfrac{p_1+p_2}{\theta_1+\theta_2} - p_1\right| = \tfrac{1}{2^k}\left|\theta_2\tfrac{p_1+p_2}{\theta_1+\theta_2} - p_2\right| . \tag{9}$$

- Consequently, the task of finding a pair of scaled dyadic approximations is now reduced to finding a single approximation:

$$\boxed{\theta^* \approx p/q} \quad \text{where} \quad \theta^* = \tfrac{\theta_1}{\theta_1+\theta_2}, p = p_1, q = p_1 + p_2 \;\; !!!$$

Approximation of pair of constants

- Main result for m=2:

Theorem 1. *Let* θ_1, θ_2 *be irrational numbers of the same sign. Then, there exist infinitely many integers* k *and real numbers* ξ*, such that*

$$\begin{aligned}\Delta_\xi(k) &= \frac{1}{\xi}\min_{p_1,p_2}\max\left\{\left|\theta_1\xi - p_1/2^k\right|, \left|\theta_2\xi - p_2/2^k\right|\right\} \\ &< \kappa\left(\frac{\theta_1}{\theta_1+\theta_2}\right)\frac{4}{|\theta_1+\theta_2|}2^{-2k} = O\left(2^{-2k}\right). \qquad (16)\end{aligned}$$

Extension to m-ary case

- Our main result:

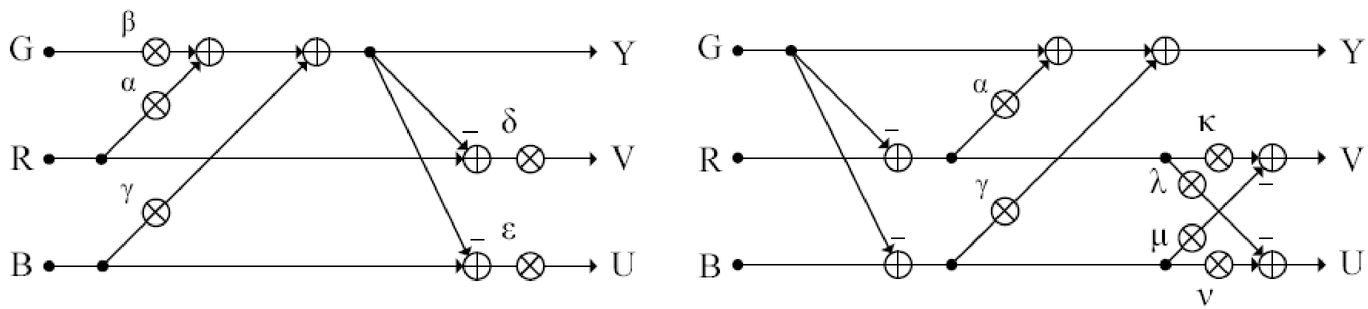
Theorem 2. *Let $\theta_1, \ldots, \theta_2$ be $m > 2$ irrational numbers of the same sign. Then, there exist infinitely many integers k and real values ξ, such that*

$$\begin{aligned} \Delta_\xi(k) &= \frac{1}{\xi} \min_{p_1,\ldots,p_m} \max_i \left\{ \left| \theta_i \xi - p_i / 2^k \right| \right\} \\ &< \frac{m-1}{m} \left(\min_{ij} \{ |\theta_i + \theta_j| \} \right)^{-\frac{1}{m-1}} 2^{-(k-1)\left(1+\frac{1}{m-1}\right)} \\ &= O\left(2^{-k\left(1+\frac{1}{m-1}\right)} \right) . \qquad (26) \end{aligned}$$

- Idea of proof: we scan indices *i,j=1..m*, apply Lemma 1 to balance errors, and find best rational approximations for remaining *m-1* constants.

Applications to design of color transforms

- Example factorizations and transforms:



Factors	Color spaces / Standards					
	YUV PAL	**YDbDr** SECAM	**YPbPr/YCbCr** ITU-R BT.601	**YPbPr/YCbCr** ITU-R BT.709	**YIQ** NTSC	**YSrSb** [6]
α	0.299	0.299	0.299	0.2125	0.299	0.3227
β	0.587	0.587	0.587	0.7154	0.587	0.3447
γ	0.114	0.114	0.114	0.0721	0.114	0.3326
δ	$0.615\,\frac{1}{1-\alpha}$	$1.333\,\frac{1}{1-\alpha}$	$\frac{1}{2}\frac{1}{1-\alpha}$	$\frac{1}{2}\frac{1}{1-\alpha}$		
ϵ	$0.436\,\frac{1}{1-\gamma}$	$-1.333\,\frac{1}{1-\gamma}$	$\frac{1}{2}\frac{1}{1-\gamma}$	$\frac{1}{2}\frac{1}{1-\gamma}$		
κ	0.615	1.333	$\frac{1}{2}$	$\frac{1}{2}$	$0.877\,(1-\alpha)\cos(33)$ $+0.492\,\alpha\,\sin(33)$	-0.1643
λ	$0.436\,\frac{\alpha}{1-\gamma}$	$-1.333\,\frac{\alpha}{1-\gamma}$	$\frac{1}{2}\frac{\alpha}{1-\gamma}$	$\frac{1}{2}\frac{\alpha}{1-\gamma}$	$-0.877\,(1-\alpha)\sin(33)$ $+0.492\,\alpha\,\cos(33)$	0.5095
μ	$0.615\,\frac{\gamma}{1-\alpha}$	$1.333\,\frac{\gamma}{1-\alpha}$	$\frac{1}{2}\frac{\gamma}{1-\alpha}$	$\frac{1}{2}\frac{\gamma}{1-\alpha}$	$0.877\,\gamma\,\cos(33)$ $+0.492\,(1-\gamma)\sin(33)$	0.3470
ν	0.436	-1.333	$\frac{1}{2}$	$\frac{1}{2}$	$-0.877\,\gamma\,\sin(33)$ $+0.492\,(1-\gamma)\cos(33)$	0.3870

Applications: CbCr factors

- Direct vs. scaled approximations:

Table 2. Approximations of a pair of constants $\theta_1 = \gamma(\text{YCbCr}) \approx 0.5643340858$, and $\theta_2 = \delta(\text{YCbCr}) \approx 0.7132667618$

Direct dyadic approximations: $\theta_1 \approx p_1/2^k, \quad \theta_2 \approx p_2/2^k$				Associated rational appr-s: $\theta^* = \theta_1/(\theta_1+\theta_2) \approx p/q$			Scaled dyadic approximations: $\theta_1\xi^* \approx p_1/2^k, \quad \theta_2\xi^* \approx p_2/2^k$			
k	p_1	p_2	$\max_i \left\lvert\theta_i - \frac{p_i}{2^k}\right\rvert$	q	p	$\left\lvert\theta^* - \frac{p}{q}\right\rvert$	$\xi^* = \frac{1}{2^k}\frac{q}{\theta_1+\theta_2}$	p_1	p_2	$\frac{1}{\xi^*}\max_i \left\lvert\theta_i\xi^* - \frac{p_i}{2^k}\right\rvert$
1	1	1	0.2132667618	2	1	0.0582860744	0.7827170762	1	1	0.0744663380
2	2	3	0.0643340858							
3	5	6	0.0606659142	9	4	0.0027305188	0.8805567108	4	5	0.0030718336
4	9	11	0.0257667618							
5	18	23	0.0054832382	43	19	0.0001465395	1.0517760712	19	24	0.0001872190
6	36	46	0.0054832382							
7	72	91	0.0023292618	163	72	0.0000038658	0.9967412768	72	91	0.0000049389
8	144	183	0.0018340858							
9	289	365	0.0003761368							
10	578	730	0.0003761368							
11	1156	1461	0.0001190392							

- Notable example:
  - k=3: (1/2, 5/8) ; ~ 20 times more precise than non-scaled approximation

Applications: YUV luminance factors

- Direct vs. scaled approximations:

Table 3. Approximations of constants: $\theta_1 = \alpha = 0.299$, $\theta_2 = \beta = 0.587$, and $\theta_3 = \gamma = 0.114$.

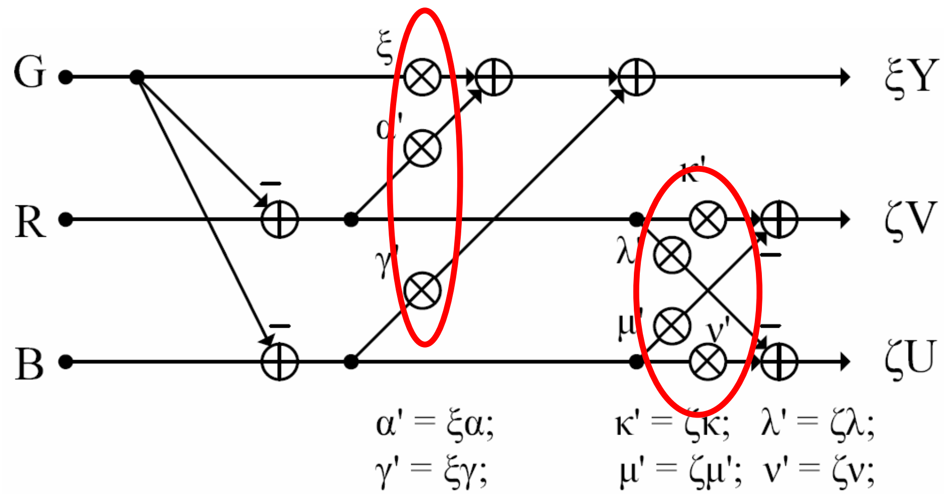
Direct dyadic approximations: $\theta_1 \approx p_1/2^k$, $\theta_2 \approx p_2/2^k$, $\theta_3 \approx p_3/2^k$					Associated rational appr-s: $\theta^*_{ij} = \theta_i/(\theta_i+\theta_j) \approx p/q$					Scaled dyadic approximations: $\theta_1\xi^* \approx p_1/2^k$, $\theta_2\xi^* \approx p_2/2^k$, $\theta_3\xi^* \approx p_3/2^k$				
k	p_1	p_2	p_3	$\max_i \left\|\theta_i - \frac{p_i}{2^k}\right\|$	i	j	q	p	$\left\|\theta^*_{ij} - \frac{p}{q}\right\|$	$\xi^* = \frac{1}{2^k}\frac{q}{\theta_i+\theta_j}$	p_1	p_2	p_3	$\frac{1}{\xi^*}\max_i \left\|\theta_i\xi^* - \frac{p_i}{2^k}\right\|$
1	1	1	0	0.2010000000										
2	1	2	0	0.1140000000										
3	2	5	1	0.0490000000										
4	5	9	2	0.0245000000	1	3	7	5	0.00968523	1.0593220339	5	10	2	0.0040000000
5	10	19	4	0.0135000000	2	3	19	8	0.00548089	0.8470042796	8	16	3	0.0038421053
6	19	38	7	0.0067500000										
7	38	75	15	0.0031875000										
8	77	150	29	0.0017812500	1	3	76	55	0.00028673	0.7188256659	55	108	21	0.0001184211
9	153	301	58	0.0008906250										
10	306	601	117	0.0002578125	1	3	413	299	0	0.9765625000	299	587	114	0
11	612	1202	233	0.0002304688										

- Notable example:
  - k=4: (5/16, 5/8, 1/8) ;

$$Y = \alpha R + \beta G + \gamma B \;\rightsquigarrow\; \left[\begin{array}{rcl} x & = & G + (R \gg 1); \\ y & = & (x + B) \gg 3; \\ Y' & = & y + (x \gg 1); \end{array}\right.$$

Applications: scaled YUV,YIQ, etc.

- Generalized flow-graph with added scale factors



- Separate groups of factors leading to luminance (Y) and chrominance (U,V) outputs
- Can also be used hybrid logic design (when same module is to be used for support of multiple color spaces)

Conclusions

- Done:
  - proposed a technique for improving implementations of color transforms by introducing scale factors
  - studied the underlying approximation problem
    - established its connection to Diophantine approximations
    - derived precision bounds
  - shown several examples of how it can be used to optimize implementations of color transforms
- Future work
  - consider other color spaces;
  - consider other application of this technique